# ON THE TRANSFORMATION OF CONVEX POINT SETS.*

BY J. L. WALSH.

It is the primary purpose of this note to prove that a one-to-one point transformation of the plane which transforms every convex point set into a convex point set is a collineation. It is also proved that a regular plane curve which remains convex under all circular transformations of the plane is a circle or circular arc. These results are then extended to space of three dimensions. Throughout the paper we consider geometry in the real domain, but as Professor C. L. Bouton pointed out to me, the discussion holds with minor changes for complex geometry.

§ 1. A point set is said to be *convex* when and only when any two points of the set are joined by a line segment (finite or infinite) consisting entirely of points of the set. According to this definition we shall prove

THEOREM I. *A necessary and sufficient condition that a one-to-one point transformation of the plane transform every convex point set into a convex point set is that it be a collineation.*

This theorem tacitly assumes the convention, usual in the geometry of the collineation, of a line of points at infinity.† The naturalness of this convention in the present case will appear later.

The sufficiency of the condition of Theorem I is obvious. A collineation transforms lines into lines, line segments into line segments, and hence every convex point set into a convex point set. We proceed to prove the necessity of the condition.

If any two points of the plane are denoted by $A'$ and $B'$, which are the transforms by the transformation $T$ considered of points $A$ and $B$ respectively, we shall prove that any point $X'$ collinear with $A'$ and $B'$ is the transform by $T$ of a point $X$ collinear with $A$ and $B$. For convenience in phraseology suppose that $A$, $B$, $A'$, and $B'$ are finite points. Suppose $X'$, collinear with $A'$ and $B'$, to be the transform by $T$ of a point $X$ not collinear with $A$ and $B$. The finite segment $AB$ is transformed by $T$ into a point set which includes all points of either the finite or the infinite

* Presented to the American Mathematical Society, September, 1920.

† Some writers define a point set as convex when and only when the finite line segment which joins any two points of the set consists wholly of points of the set. According to this definition, no convention is necessary regarding points at infinity; the following theorem is then analogous to Theorem I and is easily proved by the methods used in the present paper:

*A necessary and sufficient condition that a one-to-one point transformation of the plane transform every convex point set into a convex point set is that it be an affine transformation.*

segment $A'B'$, and $X'$ cannot lie on that particular segment $A'B'$. Hence it is possible to determine a point $Y$ on the finite segment $AB$ such that $Y$ is transformed by $T$ into a point $Y'$ collinear with $A'$ and $B'$, and such that $A'$ and $B'$ separate $X'$ and $Y'$. Consider now the finite segment $XY$, a convex point set. This is transformed by $T$ into a point set which includes neither $A'$ nor $B'$, which can include neither the finite nor infinite segment $X'Y'$, and hence which is not convex.

We have thus proved that any three collinear points $A'$, $B'$, $X'$ are the transforms by $T$ of three collinear points $A$, $B$, $X$. It follows immediately from the theorem of Darboux cited below that the inverse of $T$ is a collineation, so $T$ itself is a collineation and the proof of Theorem I is complete.

We have used here the following theorem:

*A one-to-one point transformation of the plane or of space such that collinear points in one configuration correspond to collinear points in the other configuration is a collineation.**

Our convention of a line of points at infinity now appears as entirely natural in connection with Theorem I and hence in the definition of convexity. Without a similar convention, it is not true that all collineations are one-to-one transformations. Without our convention, it is not true that all collineations transform straight lines into straight lines; without it the theorem of Darboux is no longer true. We have therefore introduced this convention to avoid the long circumlocutions which would otherwise be necessary.

It is worth while to notice that the proof of the necessity of the condition of Theorem I uses merely the fact that $T$ transforms every line segment into a convex point set; a correspondingly more general statement of the theorem can be made.

§ 2. Instead of considering convex point sets as in § 1, we shall now consider convex curves. A curve (whether closed or not) is said to be *convex* when and only when it is regular† and no three points of the curve

* Darboux, Mathematische Annalen, vol. XVII (1880), pp. 55–61; p. 59. See also Swift, Bulletin of the American Mathematical Society, (2) vol. X (1903–1904), pp. 247, 361.

Under the hypothesis of this theorem it is easy to prove that an entire straight line in either configuration corresponds to an entire straight line in the other configuration. Darboux pointed out that it was not necessary to suppose the transformation continuous in order to prove it a collineation.

We state explicitly that although the theorem of Darboux and the necessity of the condition of Theorem I are both concerned with the nature of a transformation, neither theorem makes any explicit hypothesis regarding the inverse transformation except its existence and one-to-one-ness.

† For the definition of a regular curve, see Osgood, Funktionentheorie, p. 51. The definition of a regular surface is precisely analogous.

When we consider (as here) transformations of the plane which do not transform every infinite point into an infinite point, the definition of a regular curve is to be revised so that every regular

are collinear unless they lie on a line segment which is an arc of the curve. We shall prove

THEOREM II. *A necessary and sufficient condition that a one-to-one point transformation of the plane transform every convex curve into a convex curve is that it be a collineation.*

A collineation transforms straight lines into straight lines, line segments into line segments, and hence every convex curve into a convex curve. The sufficiency of the condition is thus obvious. It will be noticed that the necessity of the condition does not follow from Theorem I or from the theorem indicated at the close of § 1. And neither of those results follows from the necessity of the condition of Theorem II; different proofs are necessary. We shall show that a transformation $T$ which transforms every convex curve into a convex curve transforms every straight line into a straight line or a line segment.

Suppose a straight line $C$ to be transformed by $T$ into a curve $C'$ other than a straight line or line segment. Choose a line $l'$ which has two points $M'$ and $N'$ in common with $C'$ yet which do not lie on a segment of $l'$ which is an arc of $C'$. Such a choice is evidently possible, since $C'$ is a regular curve. Let $P'$ be any point of $l'$ not on $C'$, and denote by $P$ the point which is transformed into $P'$ by $T$. Denote by $M$ and $N$ the points which are transformed by $T$ into $M'$ and $N'$. The curve composed of the segments $PM$ and $MN$ is convex, but is transformed by $T$ into a curve which is not convex.

We ave proved that $T$ transforms every line into a line or line segment. Then $T$ transforms collinear points into collinear points and hence is a collineation.

§ 3. It is not uninteresting to consider groups of transformations other than collineations, and to determine what convex curves remain convex under all transformations of the group. We shall prove:

THEOREM III. *A necessary and sufficient condition that a convex curve remain convex under all circular transformations is that it be a circle or circular arc.**

The sufficiency of the condition is obvious—every circle or circular arc is transformed into a circle or circular arc, which is convex. To prove the necessity of the condition, let us choose three points on the curve $C$ considered which do not lie on a circular arc that is part of $C$. Such

curve is transformed by a collineation into a regular curve. Cf. Osgood, l.c., pp. 324, 150. The definition of a regular surface is of course to be revised accordingly.

Doubtless Theorems II, III, V, and VI remain true if in the definition of convexity the requirement of regularity is replaced by a suitable less stringent requirement.

* The term *circle* here includes straight line, *circular arc* includes segment of a line. The corresponding convention is not made for the sphere and plane.

choice is possible when and only when $C$ itself is not a circle or circular arc. Transform the circle through these three points into a straight line by means of a circular transformation, none of the three points being transformed to infinity. Then $C$ is transformed into a curve that is evidently not convex.

§ 4. It is ordinarily not difficult to determine what curves are transformed into convex curves by a *single* transformation instead of a *group* of transformations. Aside from regularity, the condition for a curve $C$ is merely that $C$ shall not be cut by any curve $l$ in more than two points not lying on an arc of $l$ that is part of $C$, where $l$ is any curve corresponding under the transformation to a straight line. A necessary and sufficient condition that a regular curve be transformed by an inversion into a convex curve is that no three points of it distinct from the center of inversion be concyclic with the center of inversion, unless these three points lie on an arc of the curve which is also an arc of a circle through the center of inversion.

§ 5. We shall now generalize the preceding results to three-dimensional space. If we make the usual convention of a plane at infinity, then the definition of a convex point set, the theorem of Darboux, and the proof of Theorem I hold without change for space. Hence we have

THEOREM IV. *A necessary and sufficient condition that a one-to-one point transformation of space transform every convex point set into a convex point set is that it be a collineation.*

§ 6. We shall now prove the space analogue of Theorem II:

THEOREM V. *A necessary and sufficient condition that a one-to-one point transformation of space transform every convex surface into a convex surface is that it be a collineation.*

A surface, whether closed or not, is said to be *convex* when and only when it is regular and no three points of the surface are collinear unless they lie on a line segment every point of which is a point of the surface. The sufficiency of the condition of Theorem V is therefore evident. We proceed to demonstrate its necessity.

If a one-to-one point transformation $T$ of space transforms every convex surface into a convex surface, then it transforms every plane into a plane or a portion of a plane. For suppose a plane $\pi$ is transformed by $T$ into a surface $\pi'$ not a plane nor a portion of a plane. We can cut $\pi'$ by a line $\lambda'$ in two points $M'$, $N'$ which do not lie on a segment of $\lambda'$ which lies wholly in $\pi'$. Choose a point $P'$ on $\lambda'$ but not on $\pi'$. Denote by $M$, $N$, $P$ the points which are transformed by $T$ into $M'$, $N'$, $P'$ respectively. Let $\sigma$ be any plane through $P$ intersecting $\pi$ in a line $\nu$ so that $\nu$ does not separate $M$ from $N$. Then that half of $\sigma$ which contains $P$ and

is bounded by $\nu$ together with the half of $\pi$ which is bounded by $\nu$ and contains $M$ and $N$, forms a convex surface. This convex surface is transformed by $T$ into a surface which is not convex.

We have proved that $T$ transforms every plane into a plane or portion of a plane. Then $T$ transforms every line into a line or portion of a line, transforms collinear points into collinear points, and hence $T$ is a collineation.

§ 7. Theorem III for the plane is analogous to the following theorem for space:

THEOREM VI. *A necessary and sufficient condition that a convex surface be transformed into a convex surface by every spherical transformation is that one of these transformations transform it into a plane or half-plane.*

An alternative statement of the condition is that the surface be a sphere, spherical zone bounded by a single circle, plane, half-plane, or circular disk.

Any three points of the surface determine a circle, and this circle or an arc of it containing the three points must lie entirely in the surface. Otherwise by an inversion in space we should be able to transform a fourth point of that circle to infinity; the circle would be transformed into a straight line and the surface into a surface not convex. We now transform to infinity one point of the surface, a boundary point if the surface is not closed. Every line two finite points of which lie in the surface must have in the surface an entire segment containing these two points. Then at least part of the surface is a plane $\pi$ or a portion of a plane $\pi$. Moreover no finite point $P$ outside of $\pi$ can be a point of the surface, for then our surface would have to contain all points of a segment connecting $P$ with any point of that part of the surface which lies in the plane $\pi$.

If our surface is not an entire plane, its boundary must be a straight line. Otherwise we can cut that boundary by a straight line in two finite points and an infinite point, which three points are readily inverted into three finite points on a line yet belonging to no segment of that line entirely lying in the surface corresponding. This completes the proof.*

* In § 7 there is practically all the material needed to prove the theorem:

*If every plane section of a surface is a circle or an arc of a circle, the surface is either a sphere or a spherical zone bounded by a single circle.*

For three points of the surface determine a plane and a circle in that plane, so any three points of the surface lie on a circle or circular arc entirely in the surface.

HARVARD UNIVERSITY,
June, 1920.